AP Calculus AB
WS – Exponential Growth and Decay

Name Key
Date ______ **Period** ______

1. Half-Life. The radioactive decay of Sm-151 (an isotope of samarium) can be modeled by the differential equation $\frac{dy}{dt} = -0.0077y$, where t is measured in years. Find the half-life of Sm-151.

$y = Ce^{-.0077t}$

$\frac{1}{2} = 1e^{-.0077t}$

$\ln\frac{1}{2} = -.0077t$

$\boxed{t = 90.019 \text{ yrs.}}$

2. Half-Life. An isotope of neptunium (Np-240) has a half-life of 65 minutes. If the decay of Mp-240 is modeled by the differential equation $\frac{dy}{dt} = -ky$, where t is measured in minutes, what is the decay constant k?

$y = Ce^{-kt}$

$\frac{1}{2} = 1e^{-k(65)}$

$\ln\frac{1}{2} = -65k$

$\boxed{k = .011}$

3. Growth of Cholera bacteria. Suppose that the cholera bacteria in a colony grows unchecked according to the Law of Exponential Change. The colony starts with 1 bacterium and doubles in number every half hour.

a. How many bacteria will the colony contain at the end of 24 h?

$2 = 1e^{k(.5)}$ $\quad k = 2\ln 2 \quad y = e^{2\ln 2 \cdot t} \quad y = e^{2\ln 2(24)}$

$\ln 2 = .5k$ $\quad \approx \boxed{2.815 \times 10^{14} \text{ bacteria}}$

b. Use part a to explain why a person who feels well in the morning may be dangerously ill by evening even though, in an infected person, many bacteria are destroyed.

The bacteria grow extremely rapidly & cannot be destroyed as quickly

4. Bacteria growth. A colony of bacteria is grown under ideal conditions in a laboratory so that the population increases exponentially with time. At the end of 3 h there are 10,000 bacteria. At the end of 5 h there are 40,000 bacteria. How many bacteria were present initially?

$(0, -)$ $(3, 10{,}000)$ $(5, 40{,}000)$

$y = Ce^{kt}$

$10{,}000 = Ce^{k(3)}$

$C = \frac{10{,}000}{e^{3k}}$

$40{,}000 = Ce^{k(5)}$

$40{,}000 = \frac{10{,}000}{e^{3k}} \cdot e^{5k}$

$40{,}000 = 10{,}000e^{2k}$

$4 = e^{2k}$

$\ln 4 = 2k$

$k = \frac{1}{2}\ln 4$

$C = \frac{10{,}000}{e^{3 \cdot \frac{1}{2}\ln 4}}$

$= 1250$

$\boxed{1250 \text{ bacteria}}$

5. Radon-222. The decay equation for radon-222 gas is known to be $y = y_0 e^{-0.18t}$, with t in days. About how long will it take the amount of radon in a sealed sample of air to decay to 90% of its original value?

$y = y_0 e^{-.18t}$

$.90 = e^{-.18t}$

$\ln(.9) = -.18t$

$\boxed{t \approx .585 \text{ days}}$

8. When the valve at the bottom of a cylindrical tank is opened, the depth of liquid in the tank drops at a rage proportional to the square root of the depth of liquid (Torricelli's Law). If y(t) is the liquid's depth t minutes after the valve is opened, Torricelli's Law can be expressed as the differential equation $y' = -k\sqrt{y}$, where k is some positive constant.

a. Does the water level drop faster when a tank is full or when it is half full? Explain.

$\int \frac{dy}{y^{1/2}} \int -k \cdot dt$

$2y^{1/2} = -kt + C$

$4y = (-kt + C)^2$

The rate of change of the water level in the tank, y', has a larger absolute value when y is larger. Therefore, the water level drops faster when the tank is full.

b. Verify by differentiation that, for any constants C and k, the function $y(t) = \frac{(C-kt)^2}{4}$ is a solution of Torricelli's differential equation $y' = -k\sqrt{y}$. It's OK to assume that C – kt ≥ 0.

c. Suppose that for a certain tank, y(0) = 9 feet and y(20) = 4 feet. Find an equation for y(t). Show that the tank takes 60 minutes to empty entirely.

$y = \frac{(C-kt)^2}{4}$

$9 = \frac{(C-k\cdot 0)^2}{4}$

$36 = C^2$

$C = 6$

$y = \frac{(6-kt)^2}{4}$

$4 = \frac{(6-20k)^2}{4}$

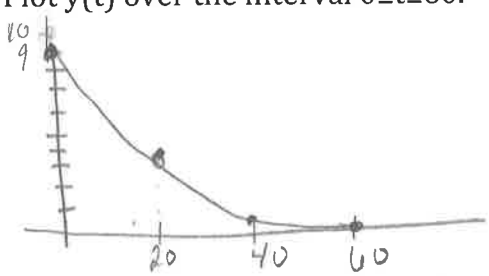
$4 = 6 - 20k$

$\frac{1}{10} = k$

$\boxed{y(t) = \frac{(6 - \frac{1}{10}t)^2}{4}}$

$y(60) = \frac{(6 \cdot \frac{1}{10} \cdot 60)^2}{4} = 0$

d. Plot y(t) over the interval 0≤t≤60.



e. At what time is the water level dropping at a rate of 0.1 feet per minute?

$-.1 = -k \cdot \sqrt{y}$

$-.1 = -\frac{1}{10}\sqrt{y}$

$+1 = y$

$1 = \frac{(6 - \frac{1}{10}t)^2}{4}$

$4 = (6 - \frac{1}{10}t)^2$

$2 = 6 - \frac{1}{10}t$

$-4 = -\frac{1}{10}t$

$t = 40$

$\boxed{t = 40 \text{ minutes}}$

6. Cooling Soup. Suppose that a cup of soup cooled from 90° C to 60° C in 10 min in a room whose temperature was 20° C. Use Newton's Law of Cooling to answer the following questions.

$(0,90)$ $(10,60)$ $(\_\_, 35)$

a. How much longer would it take the soup to cool to 35°C?

$y = A + Ce^{kt}$

$y = 20 + Ce^{kt}$

$90 = 20 + Ce^{k(0)}$

$70 = C$

$y = 20 + 70e^{kt}$

$60 = 20 + 70e^{k(10)}$

$\frac{4}{7} = e^{10k}$

$\ln\frac{4}{7} = 10k$

$k = \frac{1}{10}\ln\frac{4}{7}$

$y = 20 + 70e^{\frac{1}{10}\ln\frac{4}{7}\cdot t}$

$35 = 20 + 70e^{\frac{1}{10}\ln\frac{4}{7}t}$

$\ln\frac{3}{14} = \frac{1}{10}\ln\frac{4}{7}t$

$t = 27.527$

-10

$\boxed{17.527 \text{ min.}}$

* b. Instead of being left to stand in the room, the cup of 90°C soup is put into a freezer whose temperature is -15°C. How long will it take the soup to cool from 90°C to 35°C?

A

$y = -15 + Ce^{kt}$

$90 = -15 + Ce^{k(0)}$

$105 = C$

$35 = -15 + 105e^{-\frac{1}{10}\ln\frac{4}{7}t}$

$50 = 105e^{-\frac{1}{10}\ln\frac{4}{7}t}$

$\ln\frac{10}{21} = -\frac{1}{10}\ln\frac{4}{7}t$

$\boxed{t = 13.258 \text{ min.}}$

7. Cooling Silver. The temperature of an ingot of silver is 60°C above room temperature right now. Twenty minutes ago, it was 70°C above room temperature. How far above room temperature will the silver be

$(0, 70+A)$ $(20, 60+A)$

a. 15 minutes from now?

$y = A + Ce^{kt}$

$70 + \cancel{A} = \cancel{A} + Ce^{k(0)}$

$C = 70$

$y = A + 70e^{kt}$

$60 + \cancel{A} = \cancel{A} + 70e^{k(20)}$

$\frac{1}{20}\ln\frac{6}{7} = k$

$y = A + 70e^{\frac{1}{20}\ln\frac{6}{7}t}$

$y = A + 70e^{\frac{1}{20}\ln\frac{6}{7}\cdot 35}$

$y = A + 53.449$

$\boxed{53.449° \text{ above room temp}}$

b. 2 hours from now?

$y = A + 70e^{\frac{1}{20}\ln\frac{6}{7}(140)}$

$= A + 23.794$

$\boxed{23.794°C \text{ above room temp}}$

c. When will the silver be 10°C above room temperature?

$10 + \cancel{A} = \cancel{A} + 70e^{\frac{1}{20}\ln\frac{6}{7}t}$

$\ln\frac{1}{7} = \frac{1}{20}\ln\frac{6}{7}\cdot t$

$t = 252.469$ min.

-20

232.469 min

3 hr. 53 min