Key

AP Calculus Differential Equation Practice Problems

Unit 6

Find the general solution of each of the following differential equations.

1. $y' = 1 + x^2 + y + x^2y$
2. $y' = \frac{x}{y^2+1}$
3. $e^y \frac{dy}{dx} = 2x$
4. $\frac{dy}{dx} = k(10 - x)$
5. $3y^2 \frac{dy}{dx} = 1$
6. $y' - xy = 0$
7. $\frac{dy}{dx} = \sqrt{1-y}$
8. $xy' = y$
9. $e^x(y' + 1) = 1$

Solve each initial value problem

10. $(x^2+1)(2yy'+2y')=x$, $y(0)=0$
11. $y' = x\sqrt{x^2+1}$ $y(0)=1$
12. $y' = \frac{x}{y}$, $y(1)=4$
13. $dy = \frac{x^2}{(1+x^3)^2}, y(0) = \frac{1}{2}$
14. $dy = y^2(1+x^2), y(0) = 1$

$$\left(-\frac{1}{2}x+c\right)\left(-\frac{1}{2}x+c\right)$$

$$\frac{1}{4}x^2-\frac{1}{2}cx-\frac{1}{2}cx+c^2$$

$$1-\left(\frac{1}{4}x^2-cx+c^2\right)$$

$$1-\frac{1}{4}x^2+cx-c^2$$

1. $\frac{dy}{dx} = (1+x^2) + (y + x^2y)$

$\frac{dy}{dx} = (1+x^2) + y(1+x^2)$

$\frac{dy}{dx} = (1+x^2)(1+y)$

$\int \frac{1}{1+y}\,dy = \int 1+x^2\,dx$

$\ln|1+y| = x + \frac{1}{3}x^3 + C$

$e^{\ln|1+y|} = e^{x+\frac{1}{3}x^3} \cdot e^c$

$1+y = Ce^{x+\frac{1}{3}x^3}$

$\boxed{y = Ce^{x+\frac{1}{3}x^3} - 1}$

2. $\frac{dy}{dx} = \frac{x}{y^2+1}$

$\int y^2+1\,dy = \int x\,dx$

$\boxed{\frac{1}{3}y^3 + y = \frac{1}{2}x^2 + C}$

3. $e^y \frac{dy}{dx} = 2x$

$\int e^y dy = \int 2x\,dx$

$e^y = x^2 + C$

$\ln e^y = \ln|x^2+c|$

$\boxed{y = \ln|x^2+c|}$

4. $\frac{dy}{dx} = K(10-x)$

$\int dy = \int K(10-x)\,dx$

$y = K\int 10-x\,dx$

$y = K\left[10x - \frac{1}{2}x^2 + C\right]$

$\boxed{y = 10Kx - \frac{1}{2}Kx^2 + C}$

5. $3y^2\frac{dy}{dx} = 1$

$\int 3y^2\,dy = \int 1\,dx$

$y^3 = x + C$

$\boxed{y = \sqrt[3]{x+C}}$

6. $y' - xy = 0$

$\frac{dy}{dx} = xy$

$\int \frac{1}{y}\,dy = \int x\,dx$

$\ln y = \frac{1}{2}x^2 + C$

$e^{\ln y} = e^{\frac{1}{2}x^2 + C}$

$y = e^{\frac{1}{2}x^2} \cdot e^{C}$

$\boxed{y = Ce^{\frac{1}{2}x^2}}$

7. $\frac{dy}{dx} = \sqrt{1-y}$

$$\int \frac{1}{\sqrt{1-y}}\,dy = \int 1\,dx$$

$u = 1-y$

$du = -dy$

$dy = -du$

$$\int \frac{1}{\sqrt{u}} - du = x + C$$

$$-\int u^{-\frac{1}{2}} du = x + C$$

$$-2u^{\frac{1}{2}} = x + C$$

$$-2\sqrt{1-y} = x + C$$

$$\left(\sqrt{1-y}\right)^2 = \left(-\tfrac{1}{2}x + C\right)^2$$

$$1-y = \left(-\tfrac{1}{2}x + C\right)^2$$

$$1 - \left(-\tfrac{1}{2}x + C\right)^2 = y$$

$$\boxed{y = 1 - \left(-\tfrac{1}{2}x + C\right)^2}$$

8. $xy' = y$

$$x\frac{dy}{dx} = y$$

$$x\,dy = y\,dx$$

$$\frac{1}{y}dy = \frac{1}{x}dx$$

$$\ln y = \ln x + C$$

$$e^{\ln y} = e^{\ln x + C}$$

$$y = e^{\ln x} \cdot e^{c}$$

$$\boxed{y = Cx}$$

9. $e^{x}(y'+1) = 1$

$$y' + 1 = \frac{1}{e^{x}}$$

$$\frac{dy}{dx} = e^{-x} - 1$$

$$\int dy = \int e^{-x} - 1\,dx$$

$$\boxed{y = -e^{-x} - x + C}$$

10. $(x^2+1)(2yy'+2y')=x \qquad y(0)=0$

$(x^2+1)(2y+2)\frac{dy}{dx}=x$

$\int 2y+2\,dy=\int \frac{x}{x^2+1}\,dx \qquad u=x^2+1 \quad du=2x\,dx$

$y^2+2y=\int \frac{x}{u}\cdot\frac{du}{2x}$

$y^2+2y=\frac{1}{2}\int \frac{1}{u}\,du$

$y^2+2y=\frac{1}{2}\ln|u|+C$

$y^2+2y=\frac{1}{2}\ln|x^2+1|+C$

$0^2+2(0)=\frac{1}{2}\ln|0^2+1|+C$

$0=\frac{1}{2}\ln 1+C$

$\boxed{C=0}$

$\boxed{y^2+2y=\frac{1}{2}\ln|x^2+1|}$

11. $\frac{dy}{dx}=x\sqrt{x^2+1} \qquad y(0)=1$

$\int dy=\int x\sqrt{x^2+1}\,dx \qquad u=x^2+1 \quad du=2x\,dx \quad dx=\frac{du}{2x}$

$y=\int \cancel{x}\sqrt{u}\,\frac{du}{2x}$

$y=\frac{1}{2}\int u^{\frac{1}{2}}du$

$y=\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}}+C$

$y=\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C$

$1=\frac{1}{3}(0^2+1)^{\frac{3}{2}}+C$

$1=\frac{1}{3}(1)^{\frac{3}{2}}+C$

$1=\frac{1}{3}+C$

$-\frac{1}{3} \quad -\frac{1}{3}$

$C=\frac{2}{3}$

$\boxed{y=\frac{1}{3}(x^2+1)^{\frac{3}{2}}+\frac{2}{3}}$

12. $\frac{dy}{dx} = \frac{x}{y}$ $\quad y(1)=4$

$\int y\,dy = \int x\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C \longrightarrow y^2 = x^2 + 15$

$y = \sqrt{x^2+15}$

$\frac{1}{2}(4)^2 = \frac{1}{2}(1)^2 + C$

$8 = \frac{1}{2} + C$

$-\frac{1}{2} \quad -\frac{1}{2}$

$\frac{15}{2} = C$

13. $\frac{dy}{dx} = \frac{x^2}{(1+x^3)^2}$ $\quad y(0) = \frac{1}{2}$

$\int dy = \int \frac{x^2}{(1+x^3)^2}\,dx$ $\quad u = 1+x^3$

$du = 3x^2 dx$

$dx = \frac{du}{3x^2}$

$y = \int \frac{\cancel{x^2}}{u^2} \cdot \frac{du}{3\cancel{x^2}}$

$y = \frac{1}{3}\int \frac{1}{u^2}\,du$

$y = \frac{1}{3}\int u^{-2}\,du$

$y = \frac{1}{3} \cdot -1u^{-1} + C$

$y = -\frac{1}{3}(1+x^3)^{-1} + C$

$y = \frac{-1}{3(1+x^3)} + C$

$\frac{1}{2} = \frac{-1}{3(1+0^3)} + C$

$\frac{1}{2} = -\frac{1}{3} + C$

$+\frac{1}{3} \quad +\frac{1}{3}$

$\frac{3}{6} + \frac{2}{6} = C$

$\frac{5}{6} = C$

$y = \frac{-1}{3(1+x^3)} + \frac{5}{6}$

$y = \frac{-2+5(1+x^3)}{6}$

$y = \frac{-2+5+5x^3}{6} = \boxed{\frac{3+5x^3}{6}}$

14. $dy = y^2(1+x^2)$, $\quad y(0)=1$

$$\int \frac{1}{y^2}\,dy = \int (1+x^2)\,dx$$

$$\int y^{-2}\,dy = \int 1+x^2\,dx$$

$$-y^{-1} = x + \tfrac{1}{3}x^3 + C$$

$$-(1)^{-1} = 0 + \tfrac{1}{3}0^3 + C$$

$$-1 = C$$

$$-\frac{1}{y} = x + \tfrac{1}{3}x^3 - 1$$

$$y = \frac{-1}{x+\frac{1}{3}x^3-1} \cdot \frac{3}{3}$$

$$\boxed{y = \frac{-3}{x^3+3x-3}}$$