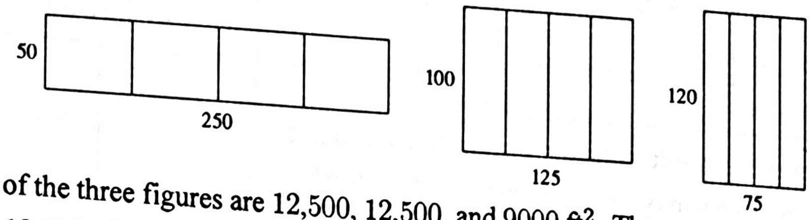
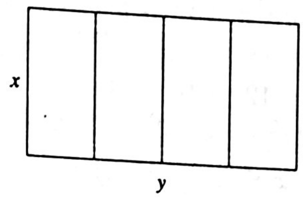
Section 4.6 Optimization Problems

1. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b)



Let x denote the length of each of two sides and three dividers. Let y denote the length of the other two sides.

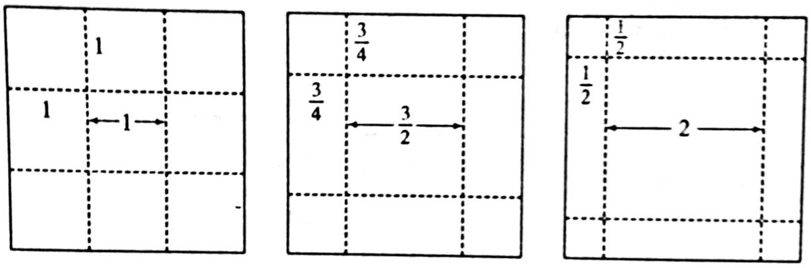
(c) Area $A =$ length $\times$ width $= y \cdot x$

(d) Length of fencing $= 750 \Longrightarrow 5x + 2y = 750$

(e) $5x + 2y = 750 \Longrightarrow y = 375 - \frac{5}{2}x \Longrightarrow A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

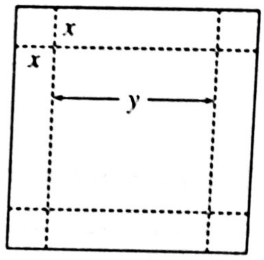
(f) $A'(x) = 375 - 5x = 0 \Longrightarrow x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5$ ft^2. These values are between the values in the first and second figures in part (a). Our original estimate was low.

2. (a)



The volumes of the resulting boxes are 1, 1.6875, and 2 ft^3. There appears to be a maximum volume of at least 2 ft^3.

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



(c) Volume V = length × width × height $\Longrightarrow$
$V = y \cdot y \cdot x = xy^2$

(d) Length of cardboard $= 3 \Longrightarrow x + y + x = 3 \Longrightarrow y + 2x = 3$

(e) $y + 2x = 3 \Longrightarrow y = 3 - 2x \Longrightarrow V(x) = x(3-2x)^2$

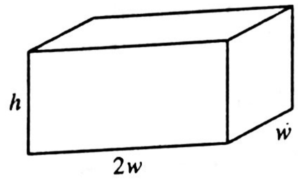
(f) $V(x) = x(3-2x)^2 = x(4x^2 - 12x + 9) = 4x^3 - 12x^2 + 9x \Longrightarrow$
$V'(x) = 12x^2 - 24x + 9 = 3(4x^2 - 8x + 3) = 3(2x-1)(2x-3)$, so the critical numbers are $x = \frac{1}{2}$ and $x = \frac{3}{2}$. Now $0 \le x \le \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is $V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2)^2 = 2$ ft^3, which is the value found from our third figure in part (a).

3. Let b be the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \Longrightarrow$ $h = (1200 - b^2)/(4b)$. The volume is $V = b^2h = b^2(1200 - b^2)/4b = 300b - b^3/4 \Longrightarrow$ $V'(b) = 300 - \frac{3}{4}b^2$. $V'(b) = 0 \Longrightarrow b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Maximum or Minimum Values. If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2h = (20)^2(10) = 4000$ cm^3.

4. Let b be the area of the base of the box and h be its height, so $32{,}000 = hb^2$ or $h = 32{,}000/b^2$. The surface area of the open box is $b^2 + 4hb = b^2 + 4(32{,}000/b^2)b = b^2 + 4(32{,}000)/b$. So $V'(b) = 2b - 4(32{,}000)/b^2 = 2(b^3 - 64{,}000)/b^2 = 0 \Longleftrightarrow b = \sqrt[3]{64{,}000} = 40$. This gives an absolute minimum since $V'(b) < 0$ if $b < 40$ and $V'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

5. (a) Let the rectangle have sides x and y and area A, so $A = xy$ or $y = A/x$. The problem is to minimize the perimeter $= 2x + 2y = 2x + 2A/x = P(x)$. Now $P'(x) = 2 - 2A/x^2 = 2(x^2 - A)/x^2$. So the critical number is $x = \sqrt{A}$. Since $P'(x) < 0$ for $0 < x < \sqrt{A}$ and $P'(x) > 0$ for $x > \sqrt{A}$, there is an absolute minimum at $x = \sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A} = \sqrt{A}$, so the rectangle is a square.

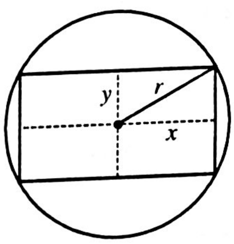
(b) Let p be the perimeter and x and y the lengths of the sides, so $p = 2x + 2y \Longrightarrow y = \frac{1}{2}p - x$. The area is $A(x) = x\left(\frac{1}{2}p - x\right) = \frac{1}{2}px - x^2$. Now $0 = A'(x) = \frac{1}{2}p - 2x \Longrightarrow x = \frac{1}{4}p$. Since $A''(x) = -2 < 0$, there is an absolute maximum where $x = \frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are $\frac{1}{4}p$ and $\frac{1}{2}p - \frac{1}{4}p = \frac{1}{4}p$, so the rectangle is a square.

6. 

$10 = (2w)(w)h = 2w^2h$, so $h = 5/w^2$. The cost is $10(2w^2) + 6[2(2wh) + 2hw] = 20w^2 + 36wh$, so $C(w) = 20w^2 + 36w(5/w^2) = 20w^2 + 180/w$. $C'(w) = 40w - 180/w^2 = 40\left(w^3 - \frac{9}{2}\right)/w^2 \Longrightarrow$ $w = \sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{9}{2}}$. $C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \frac{180}{\sqrt[3]{9/2}} \approx \$163.54$.

7. For (x, y) on the line $y = 2x - 3$, the distance to the origin is $\sqrt{(x-0)^2 + (2x-3)^2}$. We minimize the square of the distance, that is, $x^2 + (2x-3)^2 = 5x^2 - 12x + 9 = D(x)$. $D'(x) = 10x - 12 = 0 \Longrightarrow$ $x = \frac{6}{5}$. Since there is a point closest to the origin, $x = \frac{6}{5}$ and hence $y = -\frac{3}{5}$. So the point is $\left(\frac{6}{5}, -\frac{3}{5}\right)$.

8. By symmetry, the points are (x, y) and $(x, -y)$, where $y > 0$. The square of the distance is $D(x) = (x-2)^2 + y^2 = (x-2)^2 + (4 + x^2) = 2x^2 - 4x + 8$. So $D'(x) = 4x - 4 = 0 \Longrightarrow x = 1$ and $y = \pm\sqrt{4+1} = \pm\sqrt{5}$. The points are $(1, \pm\sqrt{5})$.

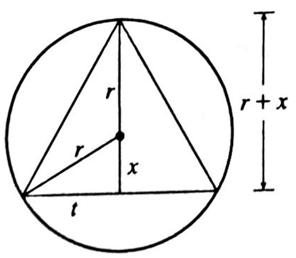
9. 

Area of rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now

$$A'(x) = 4\left(\sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}.$$

The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}r$ and $2y = \sqrt{2}r$.

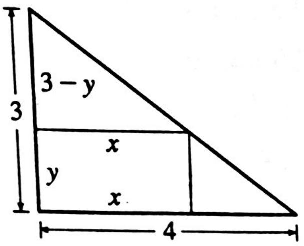
10. 

The area of the triangle is

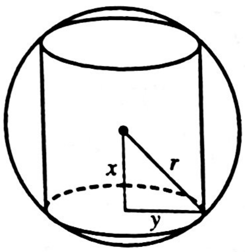
$A(x) = \frac{1}{2}(2t)(r + x) = t(r + x) = \sqrt{r^2 - x^2}\,(r + x)$. Then

$$0 = A'(x) = r\frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} + x\frac{-2x}{2\sqrt{r^2 - x^2}}$$
$$= -\frac{x^2 + rx}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \Longrightarrow$$

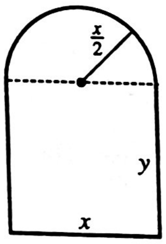
$\frac{x^2 + rx}{\sqrt{r^2 - x^2}} = \sqrt{r^2 - x^2} \Longrightarrow x^2 + rx = r^2 - x^2 \Longrightarrow 0 = 2x^2 + rx - r^2 = (2x - r)(x + r) \Longrightarrow$ $x = \frac{1}{2}r$ or $x = -r$. Now $A(r) = 0 = A(-r) \Longrightarrow$ the maximum occurs where $x = \frac{1}{2}r$, so the triangle has height $r + \frac{1}{2}r = \frac{3}{2}r$ and base $2\sqrt{r^2 - \left(\frac{1}{2}r\right)^2} = 2\sqrt{\frac{3}{4}r^2} = \sqrt{3}r$.

11. 

The rectangle has area xy. By similar triangles $\frac{3-y}{x} = \frac{3}{4} \implies$ $-4y + 12 = 3x$ or $y = -\frac{3}{4}x + 3$. So the area is $A(x) = x\left[-\frac{3}{4}x + 3\right] = -\frac{3}{4}x^2 + 3x$ where $0 \le x \le 4$. Now $0 = A'(x) = -\frac{3}{2}x + 3 \implies x = 2$ and $y = \frac{3}{2}$. Since $A(0) = A(4) = 0$, the maximum area is $A(2) = 2\left(\frac{3}{2}\right) = 3\text{ cm}^2$.

12. 

The cylinder has volume $V = \pi y^2(2x)$. Also $x^2 + y^2 = r^2 \implies$ $y^2 = r^2 - x^2$, so $V(x) = \pi\left(r^2 - x^2\right)(2x) = 2\pi\left(r^2 x - x^3\right)$, where $0 \le x \le r$. $V'(x) = 2\pi\left(r^2 - 3x^2\right) = 0 \implies x = r/\sqrt{3}$. Now $V(0) = V(r) = 0$, so there is a maximum when $x = r/\sqrt{3}$ and $V\left(r/\sqrt{3}\right) = \pi\left(r^2 - r^2/3\right)\left(2r/\sqrt{3}\right) = 4\pi r^3/\left(3\sqrt{3}\right)$.

13. 

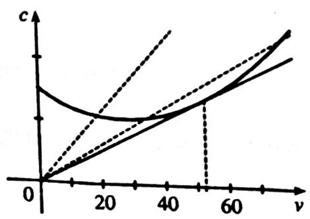
We are given $2y + x + \pi\left(\frac{x}{2}\right) = 30$, so $y = \frac{1}{2}\left[30 - x - \frac{\pi x}{2}\right]$. The area is $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so

$$A(x) = x\left[15 - \frac{x}{2} - \frac{\pi x}{4}\right] + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2.$$

$$A'(x) = 15 - \left(1 + \frac{\pi}{4}\right)x = 0 \implies x = \frac{15}{1 + \pi/4} = \frac{60}{4 + \pi}.$$

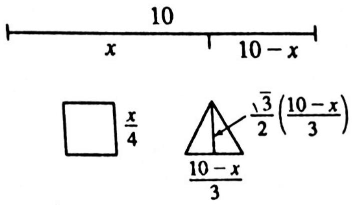
Clearly this gives a maximum, so the dimensions are $x = \frac{60}{4+\pi}$ ft and $y = 15 - \frac{30}{4+\pi} - \frac{15\pi}{4+\pi} = \frac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

14. We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\frac{c}{v} = \frac{\text{gallons/hour}}{\text{miles/hour}} = \frac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\frac{dG}{dv} = \frac{d}{dv}\left(\frac{c}{v}\right) = \frac{v\frac{dc}{dv} - c\frac{dv}{dv}}{v^2} = \frac{v\frac{dc}{dv} - c}{v^2}$.



This is 0 when $v\frac{dc}{dv} - c = 0 \iff \frac{dc}{dv} = \frac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi/h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

15.



Let x be the length of the wire used for the square. The total area is

$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)$$
$$= \tfrac{1}{16}x^2 + \tfrac{\sqrt{3}}{36}(10-x)^2,\ 0 \le x \le 10.$$

$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10-x) = 0 \iff \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \iff$

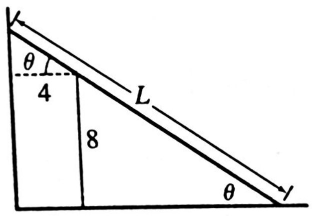
$x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$. Now $A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81$, $A(10) = \frac{100}{16} = 6.25$

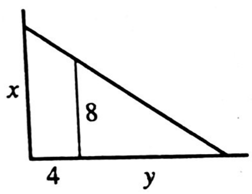
and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \approx 2.72$, so

(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}} \approx 4.35$ m.

16.





$L = 8\csc\theta + 4\sec\theta,\ 0 < \theta < \frac{\pi}{2}$,

θ = .8999

$\frac{dL}{d\theta} = -8\csc\theta\cot\theta + 4\sec\theta\tan\theta = 0$ when

θ = tan⁻¹ ∛2

$\sec\theta\tan\theta = 2\csc\theta\cot\theta \iff \tan^3\theta = 2 \iff \tan\theta = \sqrt[3]{2} \iff$

$\theta = \tan^{-1}\sqrt[3]{2}$. $dL/d\theta < 0$ when $0 < \theta < \tan^{-1}\sqrt[3]{2}$, $dL/d\theta > 0$

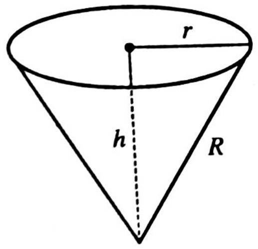
when $\tan^{-1}\sqrt[3]{2} < \theta < \frac{\pi}{2}$, so L has an absolute minimum when

$\theta = \tan^{-1}\sqrt[3]{2}$, so the shortest ladder has length

$$L = 8\frac{\sqrt{1+2^{2/3}}}{2^{1/3}} + 4\sqrt{1+2^{2/3}} \approx 16.65 \text{ ft.}$$

Another Method: Minimize $L^2 = x^2 + (4+y)^2$, where $\frac{x}{4+y} = \frac{8}{y}$.

17.



$h^2 + r^2 = R^2 \Longrightarrow$

$V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2h - h^3)$. $V'(h)$

$= \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$.

This gives an absolute maximum since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$

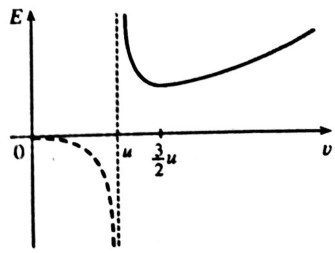
and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. Maximum volume is

$$V\left(\tfrac{1}{\sqrt{3}}R\right) = \tfrac{\pi}{3}\left(\tfrac{1}{\sqrt{3}}R^3 - \tfrac{1}{3\sqrt{3}}R^3\right) = \tfrac{2}{9\sqrt{3}}\pi R^3.$$

18. (a) $E(v) = \frac{aLv^3}{v-u} \Longrightarrow E'(v) = aL\frac{(v-u)3v^2 - v^3}{(v-u)^2} = 0$ when

$2v^3 = 3uv^2 \Longrightarrow 2v = 3u \Longrightarrow v = \frac{3}{2}u$. The First Derivative Test shows that this value of v gives the minimum value of E.

(b)

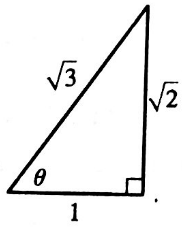


19. $S = 6sh - \frac{3}{2}s^2 \cot\theta + 3s^2\frac{\sqrt{3}}{2}\csc\theta$

(a) $\dfrac{dS}{d\theta} = \frac{3}{2}s^2\csc^2\theta - 3s^2\frac{\sqrt{3}}{2}\csc\theta\cot\theta$ or $\frac{3}{2}s^2\csc\theta\left(\csc\theta - \sqrt{3}\cot\theta\right)$.

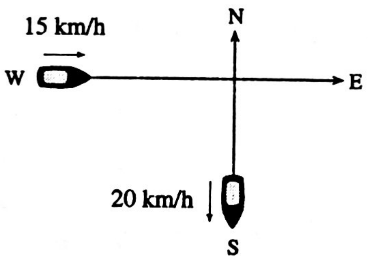
(b) $\dfrac{dS}{d\theta} = 0$ when $\csc\theta - \sqrt{3}\cot\theta = 0 \implies \dfrac{1}{\sin\theta} - \sqrt{3}\dfrac{\cos\theta}{\sin\theta} = 0 \implies \cos\theta = \frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta = \cos^{-1}\frac{1}{\sqrt{3}} \approx 55°$.

(c)



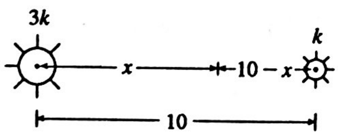
If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is $S = 6sh - \frac{3}{2}s^2\frac{1}{\sqrt{2}} + 3s^2\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{\sqrt{2}} = 6sh - \frac{3}{2\sqrt{2}}s^2 + \frac{9}{2\sqrt{2}}s^2 = 6s\left(h + \frac{1}{2\sqrt{2}}s\right)$.

20.



Let t be the time, in hours, after 2:00 P.M.. The position of the boat heading south at time t is $(0, -20t)$. The position of the boat heading east at time t is $(-15 + 15t, 0)$. If $D(t)$ is the distance between the boats at time t, we minimize
$f(t) = [D(t)]^2 = 20^2t^2 + 15^2(t-1)^2$.
$f'(t) = 800t + 450(t-1) = 1250t - 450 = 0$ when
$t = \frac{450}{1250} = 0.36$ h. $0.36\text{ h} \times \frac{60\text{ min}}{\text{h}} = 21.6\text{ min} = 21\text{ min }36\text{ s}$.
Since $f''(t) > 0$, this gives a minimum, so the boats are closest together at 2:21:36 P.M.

21.



The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(10-x)^2}$, $0 < x < 10$.

Then $I'(x) = \dfrac{-6k}{x^3} + \dfrac{2k}{(10-x)^3} = 0 \implies$

$6k(10-x)^3 = 2kx^3 \implies \sqrt[3]{3}(10-x) = x \implies$

$x = \dfrac{10\sqrt[3]{3}}{1+\sqrt[3]{3}} \approx 5.9$ ft. This gives a minimum since $I''(x) > 0$ for $0 < x < 10$.